MATH 2040 Lecture 12 (20/10/2016)

Recall: $(V, \langle , \rangle)$

$\langle , \rangle \rightsquigarrow$ length + angle ($\mathbb{F} = \mathbb{R}$)

inner product

$\|\vec{v}\| := \sqrt{\langle \vec{v}, \vec{v} \rangle}$

$\cos\theta := \dfrac{\langle \vec{v}, \vec{w} \rangle}{\|\vec{v}\| \, \|\vec{w}\|}$.

Q: $\|\cdot\|$ norm $\Longrightarrow \exists \langle , \rangle$ s.t. $\|\cdot\| = \sqrt{\langle \cdot , \cdot \rangle}$. ?

parallelogram law

Remember: $\vec{u} \perp \vec{v}$ iff $\langle \vec{u}, \vec{v} \rangle = 0$

"orthogonal"

Def$^{\underline{n}}$: $S \subseteq (V, \langle , \rangle)$ subset

- S orthogonal if $\vec{u} \perp \vec{v}$ $\forall \vec{u}, \vec{v} \in S$
- S orthonormal if orthogonal + $\|\vec{u}\| = 1$ $\forall \vec{u} \in S$

E.g.: $(\mathbb{R}^2, \cdot)$ std dot product

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \xrightarrow[\vec{v} \rightsquigarrow \frac{\vec{v}}{\|\vec{v}\|}]{\text{"normalization"}} S' = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

orthogonal — orthonormal

Questions: (1) why care about orthogonal / orthonormal ?

(2) S lin. indep. $\xrightarrow[\text{Schmidt}]{\text{Gram-}}$ S' orthogonal ?

(1) Why care?

GOAL: Is there a "best" basis for a v.space V?

No! All bases are "the same".

GOAL': Is there a "best" basis for an inner prod. sp. $(V, \langle , \rangle)$?

Yes! orthonormal basis. Why better?

Recall: $\beta \subset V$ basis $\iff$ Any $\vec{v} \in V$ can be uniquely expressed as

$\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

how to find them?

difficult! Solve system of eq$^{\underline{n}}$!

Theorem: Suppose $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ ① orthogonal, and ② none of the $\vec{v}_i = \vec{0}$.

Then, $\forall\, \vec{y} \in \operatorname{span} S$,

$$\boxed{\vec{y} = \sum_{i=1}^{k} \frac{\langle \vec{y}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i} \quad \text{— } (*)$$

$= 1$

Corollary: $\beta \subset V$ orthonormal basis

$\Rightarrow$ any $\vec{y} \in V$ is given by $\vec{y} = \sum_{i=1}^{k} \underbrace{\langle \vec{y}, \vec{v}_i \rangle \vec{v}_i}_{\operatorname{Proj}_{\vec{v}_i}(\vec{y})}$.

Proof: $\vec{y} \in \operatorname{span} S \Rightarrow \vec{y} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$

Take $\langle\, , \rangle$ on both sides with $\vec{v}_i$.

$$\langle \vec{y}, \vec{v}_i \rangle = a_1 \underbrace{\langle \vec{v}_1, \vec{v}_i \rangle}_{\overset{\shortparallel}{0}} + a_2 \underbrace{\langle \vec{v}_2, \vec{v}_i \rangle}_{\overset{\shortparallel}{0}} + \underset{\substack{\uparrow\ \text{exep} \\ a_i \langle \vec{v}_i, \vec{v}_i \rangle}}{\cdots} + a_k \underbrace{\langle \vec{v}_k, \vec{v}_i \rangle}_{\overset{\shortparallel}{0}}$$

rearrange, $\quad a_i = \dfrac{\langle \vec{y}, \vec{v}_i \rangle}{\| \vec{v}_i \|^2}$.

$\square$

Corollary: $S = \{ \underset{\neq \vec{0}}{\vec{v}_1}, \ldots, \underset{\neq \vec{0}}{\vec{v}_k} \}$ orthogonal $\Rightarrow$ S lin. indep.

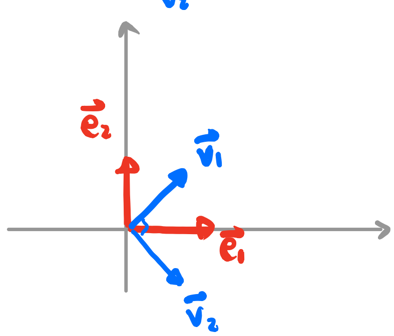
Examples (orthonormal basis)

(1) $(\mathbb{R}^n, \cdot)$ std. $\Longrightarrow$ std basis $\beta = \{ \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \}$

e.g. $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \vec{e}_1 + 2 \cdot \vec{e}_2$

$(\mathbb{C}^n, \cdot)$ std — same basis works

(2) $S' = \left\{ \underset{\vec{v}_1}{\underset{\shortparallel}{\tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}}, \underset{\vec{v}_2}{\underset{\shortparallel}{\tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}} \right\}$ O.N.B. for $\mathbb{R}^2$



$\mathbb{R}^2$

(3) $V = C([0,2\pi])$ v.s. $/ \mathbb{C}$

$$\langle f, g \rangle_{L^2} := \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} \, dx$$

$\beta = \{ f_n(x) := e^{inx} \mid n \in \mathbb{Z} \}$. "orthonormal basis"

(e.g. $f_1(x) = e^{ix} = \cos x + i \sin x$)

$$\langle f_n, f_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{inx} \cdot \overline{e^{inx}} \, dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

In fact, β is an O.N.B (in some sense)

$\because \dim_{\mathbb{C}} V = +\infty$

Any $f : [0, 2\pi] \to \mathbb{C}$, then

$$f(x) = a_0 + a_1 \sin x + b_1 \cos x + a_2 \sin 2x + b_2 \cos 2x + \cdots$$

$$= \sum_{n \in \mathbb{Z}} a_n e^{inx}$$

$\Lsh$ $a_n = \langle f(x), e^{inx} \rangle_{L^2}$.